On the spectrum of a partial theta function

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Abstract

The bivariate series $\theta(q,x) := \sum_{j=0}^{\infty} q^{j(j+1)/2}x^j$ defines a *partial theta function*. For fixed q, $\theta(q,.)$ is an entire function. We show that for $|q| \leq 0.108$ the function $\theta(q,.)$ has no multiple zeros.

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1 Introduction

Consider the bivariate series $\theta(q,x) := \sum_{j=0}^{\infty} q^{j(j+1)/2}x^j$ for $(q,x) \in \mathbf{C}^2$, $|q| < 1$. We consider x as a variable and q as a parameter. For each q fixed the series defines an entire function called a *partial theta function*. This terminology stems from the fact that $\theta(q^2, x/q) = \sum_{j=0}^{\infty} q^{j^2}x^j$ while the series $\sum_{j=-\infty}^{\infty} q^{j^2}x^j$ defines the Jacobi theta function; in the series for θ only a partial summation (i.e. excluding negative indices) is performed.

The function θ has been applied in several domains: in asymptotic analysis (see [2]), in the theory of (mock) modular forms (see [3]), in Ramanujan type q-series (see [12]), in statistical physics and combinatorics (see [11]), in questions concerning hyperbolic polynomials (i.e. real polynomials with all roots real, see [4], [9] and [7]). For more information about θ, see also [1].

For real q and x there are countably many values $0.3092493386\ldots = \tilde{q}_1 < \tilde{q}_2 < \cdots < 1$ of q such that $\theta(q,\cdot)$ has a unique multiple zero of multiplicity 2. Moreover, $\lim_{j\to\infty} \tilde{q}_j = 1$, see [9] and [7]. These values of q are said to belong to the *spectrum* of θ. The double zero is the rightmost of the real zeros of θ. For $q \in (\tilde{q}_j, \tilde{q}_{j+1}]$ the function $\theta(q,\cdot)$ has exactly j complex conjugate pairs of zeros (counted with multiplicity). (It is not clear yet whether they are all simple or not.) It is proved in [8] that $\tilde{q}_j = 1 - (\pi/2j) + o(1/j)$ and that the double real zeros of $\theta(\tilde{q}_j,.)$ tend to $-e^{\pi} = -23.1407\ldots$ (this number appears in a different context, but always in relationship with θ, in a theorem announced in [5]).

In the present paper we show that when q and x are complex, there exists a neighbourhood of 0 free of spectral points. More precisely, we prove the following theorem:

Theorem 1.1 *There is no spectral value of q for $q \in \mathbf{C}$, $|q| \leq 0.108$.*

Our interest in a neighbourhood of 0 is explained by the fact that $\theta(0,\cdot) \equiv 1$ while for any nonzero q on the open unit disk, $\theta(q,\cdot)$ is a nontrivial entire function. The number 0.108 certainly does not give the best possible estimate for the neighbourhood free from spectral values. On the other hand, it cannot be replaced by a number greater than or equal to $0.3092493386\ldots$ because the latter belongs to the spectrum of θ.

Acknowledgement. It seems that A. Sokal was the first to ask the question about the existence of a neighbourhood of 0 free of spectral values. On his homepage (see http://www.maths.qmul.ac.uk/ pjc/csgnotes/sokal/) one can find conjectures about the partial theta function. His claim that the author's result should hold for 0.2078750206 in the place of 0.108 is supported by a sketch of proof (a proof of this is reported to have been given also by Jens Forsgård, a student of B.Z. Shapiro). The author is grateful to B.Z. Shapiro for the formulation of the problem and comments on this text, and also to the anonymous referee for his useful remarks.

2 Proof of Theorem 1.1

We are looking for a function θ representable in the form of an infinite product

$$\theta = \sum_{s=0}^{\infty} q^{s(s+1)/2} x^s = \prod_{j=1}^{\infty} (1 + x/\xi_j) \ , \tag{1}$$

where $\{-\xi_j\}$ is the set of zeros of $\theta(q,\cdot)$. (For $q \in (0, 0.3092493386\ldots)$ such a presentation exists because the function $\theta(q,.)$ is an entire function of order 0 of the Laguerre-Pólya class $\mathcal{LP} - \mathcal{I}$, see [4].) For $|q|$ small enough we look for zeros of the form $-\xi_j = -1/q^j\Delta_j$ (i.e. $1/\xi_j = q^j\Delta_j$), where Δ_j are complex numbers close to 1. In other words, the zeros of θ are close to the terms of a geometric progression. When $|q|$ is small enough and all Δ_j are uniformly close to 1, then all zeros are distinct and the corresponding values of q are not from the spectrum of θ. Thus we deduce Theorem 1.1 from the following theorem whose proof follows:

Theorem 2.1 *For $|q| \leq 0.108$ the function θ is representable in the form (1) with $1/\xi_j = q^j\Delta_j$, where $\Delta_j \in [0.2118,\ 1.7882]$ for $j = 1, 2, \ldots$.*

The fact that if q and Δ_j satisfy the conditions of the theorem, then all zeros of θ are distinct, is proved at the end of Section 4.

3 Formal solution

We show first that every quantity Δ_j can be represented as a formal power series in q. To this end we observe that expanding the infinite product in (1) as a power series in x gives

$$\begin{aligned} \sum_{s=0}^{\infty} q^{s(s+1)/2} x^s &= \sum_{s=0}^{\infty} e_s(1/\xi_1, 1/\xi_2, \ldots) x^s \\ &= \sum_{s=0}^{\infty} e_s(q\Delta_1, q^2\Delta_2, \ldots) x^s \ , \end{aligned}$$

where e_s is the sth elementary symmetric function. Hence

$$e_s(\Delta_1, q\Delta_2, \ldots) = q^{s(s-1)/2} \quad , \quad s = 1, 2, \ldots \ . \tag{2}$$

We are going to show that the infinite system of these equations can be solved for Δ_1, Δ_2, ... Every equation (2) is a formal power series (FPS) in the infinitely many variables q, Δ_1, Δ_2, ... Set $\Delta^j := (\Delta_j, \Delta_{j+1}, \ldots)$, $\Delta := \Delta^1$. We denote by the letter T (indexed or not) an FPS in the indicated variables q and Δ^j. After division by q the $s = 1$ case of equations (2) reads

$$1 = \Delta_1 + q\Delta_2 + q^2\Delta_3 + \cdots = \Delta_1 + qT_{1,2}(q, \Delta^2) \tag{3}$$

The double index of $T_{1,2}$ means that this is an FPS connected with the $s = 1$ case of equations (2) and in which the first of the variables Δ_j that contributes is Δ_2. Similarly, after division by $q^{s(s+1)/2}$ equation (2) for general s becomes

$$1 = \Delta_1 \cdots \Delta_s + qT_{s,1}(q, \Delta) \tag{4}$$

In what follows we refer to equation (4) also as to equation (F_s). We denote by $(E_{1,2})$ equation (3) (written in the form $\Delta_1 = 1 - qT_{1,2}(q, \Delta^2)$) and by $(E_{2,2})$ the equation obtained from equation (F_2) after substituting in it $1 - qT_{1,2}(q, \Delta^2)$ for Δ_1. Hence equation $(E_{2,2})$ reads

$$(E_{2,2}) \ : \ 1 = (1 - qT_{1,2}(q, \Delta^2))\Delta_2 + qT_{2,1}(q, 1 - qT_{1,2}(q, \Delta^2), \Delta^2) \ .$$

Its right-hand side is of the form $\Delta_2 + qT^*(q, \Delta^2)$. Hence this can be solved for Δ_2. Indeed, one can apply the implicit function theorem here (at $q = 0$) and obtain an equation of the form

$$(E_{2,3}) \ : \ \Delta_2 = 1 - qT_{2,3}(q, \Delta^3) \ .$$

If we solve $(E_{2,3})$ for Δ_2 and substitute this in $(E_{1,2})$, we find the equation

$$(E_{1,3}) \ : \ \Delta_1 := 1 - qT_{1,3}(q, \Delta^3) \ .$$

In what follows we denote by $(E_{s,r})$ an equation of the form $\Delta_s = 1 - qT_{s,r}(q, \Delta^r)$.

The remainder of the proof proceeds by induction. Suppose that equations $(E_{j,i})$ are constructed for $j = 1, \ldots, s$, $i = j, j + 1, \ldots, s + 1$. (For $s = 2$ we already constructed equations $(E_{1,1})$, $(E_{1,2})$, $(E_{1,3})$, $(E_{2,2})$, $(E_{2,3})$.)

Consider equation (F_{s+1}). We solve the system of equations $(E_{j,s+1})$ for the variables Δ_j, $j = 1, \ldots, s$, and substitute this in (F_{s+1}). This yields an equation of the form

$$(E_{s+1,s+1}) \ : \ 1 = \tilde{\Delta}_1 \cdots \tilde{\Delta}_s \Delta_{s+1} + qT_{s+1,1}(q, \tilde{\Delta}_1, \ldots, \tilde{\Delta}_s, \Delta^{s+1}) \ ,$$

where $\tilde{\Delta}_i = 1 - qT_{i,s+1}(q, \Delta^{s+1})$. One can express Δ_{s+1} from equation $(E_{s+1,s+1})$ (the implicit function theorem is applicable at $q = 0$) which gives the equation $(E_{s+1,s+2})$. Express then Δ_{s+1} from it (i.e. set $\Delta_{s+1} = 1 - qT_{s+1,s+2}(q, \Delta^{s+2})$) and substitute $1 - qT_{s+1,s+2}(q, \Delta^{s+2})$ for Δ_{s+1} in equations $(E_{j,s+1})$, $j = 1, \ldots, s$. This gives the equations $(E_{j,s+2})$, $j = 1, \ldots, s$.

Applying this above procedure infinitely many times we obtain the equations $(E_{s,\infty})$ which express the quantities Δ_s as FPS in q of the form $\Delta_s = 1 + O(q)$. These FPS stabilize because at every substitution of Δ_s by $1 - qT_{s,r}(q, \Delta^r)$ the power of q increases.

Remark 3.1 It is easy to deduce from the above reasoning that all coefficients of Δ_s (when considered as power series in q) are integer. We list below the first 10 coefficients of Δ_1, …, Δ_5:

1	−1	−1	−1	−2	−4	−10	−25	−66	−178
1	0	0	1	3	9	24	66	180	498
1	0	0	0	0	0	−1	−3	−9	−22
1	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0

It would be interesting to (dis)prove that $\Delta_s = 1 + (-1)^s q^{\kappa_s} \Phi_s$, where Φ_s is an FPS with positive coefficients (for $s = 1$ this is proved in [11]), and the natural numbers κ_s form an increasing sequence. It is clear that $\kappa_1 = 1$, $\kappa_2 = 3$ and $\kappa_3 = 6$. It would be interesting to explicit κ_s. A combinatorial interpretation of the coefficients of Δ_1 is given in [10].

4 Proof of the convergence

Notation 4.1 We denote by U the infinite column vector whose entries equal 1 (i.e. $U = (1,\ 1,\ \dots\ )^T$) and similarly we set $V := (1,\ \Delta_1,\ \Delta_1\Delta_2,\ \Delta_1\Delta_2\Delta_3,\ \dots\ )^T$. Denote by σ_s the right-hand side of equation (2) divided by $q^{s(s+1)/2}$ (hence $\sigma_s = \Delta_1\Delta_2\dots\Delta_s + q\Delta_1\Delta_2\dots\Delta_{s-1}\Delta_{s+1} + O(q^2)$) and by L_s the infinite square matrix with 1 on the diagonal, with $q^{s-1}\Delta_s$, $q^{s-2}\Delta_s$, ..., $q\Delta_s$, 0, 0, ... on the first subdiagonal and with zeros elsewhere. That is, $L_1 = I =\mathrm{diag}(1,1,\dots)$,

$$L_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ q\Delta_2 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad , \quad L_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ q^2\Delta_3 & 1 & 0 & 0 & \cdots \\ 0 & q\Delta_3 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{etc. Hence}$$

$$L_2V = (1,\ \Delta_1 + q\Delta_2,\ \Delta_1\Delta_2,\ \Delta_1\Delta_2\Delta_3,\ \Delta_1\Delta_2\Delta_3\Delta_4,\ \dots\ )^T\ ,$$

$$L_3L_2V = (1,\ \Delta_1 + q\Delta_2 + q^2\Delta_3,\ \Delta_1\Delta_2 + q\Delta_1\Delta_3 + q^2\Delta_2\Delta_3,\ \Delta_1\Delta_2\Delta_3,\ \Delta_1\Delta_2\Delta_3\Delta_4,\ \dots\ )^T$$

and so on. It is easy to see that

$$\cdots L_4L_3L_2V = (1,\ \sigma_1,\ \sigma_2,\ \sigma_3,\ \dots\ )^T\ .$$

Indeed, if $\tilde{\sigma}_j^k$ denotes the jth elementary symmetric polynomial of the quantities Δ_1, $q\Delta_2$, ..., $q^{k-1}\Delta_k$, then for $j \leq k$

$$\tilde{\sigma}_j^k = \tilde{\sigma}_j^{k-1} + q^{k-1}\Delta_k\tilde{\sigma}_{j-1}^{k-1} \quad \text{and} \quad \tilde{\sigma}_j^k/q^{j(j-1)/2} = \tilde{\sigma}_j^{k-1}/q^{j(j-1)/2} + q^{k-j}\Delta_k(\tilde{\sigma}_{j-1}^{k-1}/q^{(j-1)(j-2)/2})\ .$$

Thus the $(j+1)$st component of the vector $L_k\cdots L_2V$ equals $\tilde{\sigma}_j^k/q^{j(j-1)/2}$. This is a polynomial in the variables q, Δ_1, ..., Δ_k. As $k \to \infty$, it stabilizes as a formal power series in the infinitely many variables q, Δ and tends to σ_j. (Stabilization is due to the increasing powers of q.) Hence the system of equations (F_s), $s = 0, 1, \dots$ (we set $(F_0) : 1 = 1$) reads

$$\cdots L_4L_3L_2V = U \quad , \quad \text{i.e.} \quad V = L_2^{-1}L_3^{-1}L_4^{-1}\cdots U\ . \tag{5}$$

We represent the matrix L_s in the form $L_s = I + N_s$ (where $(N_s)^s = 0$). In particular, $N_1 = 0$, $N_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ q\Delta_2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ etc. Hence $L_s^{-1} = I + \sum_{j=1}^{s-1}(-N_s)^j$. The following lemma is proved in the next section:

Lemma 4.2 *The entry $(L_s^{-1})_{\mu,\nu}$ of the matrix $(L_s)^{-1}$ equals*

$$\begin{cases} (-1)^{\mu-\nu}\Delta_s^{\mu-\nu}q^{(\mu-\nu)(s-\mu+1)+(\mu-\nu)(\mu-\nu-1)/2} & \text{for}\ \ \nu \leq \mu \leq s, \\ 0 & \text{otherwise.} \end{cases}$$

We are going now to justify the convergence of the formal series in q expressing the quantities Δ_j. Denote by $\beta \in (0, 0.7882]$ a number such that $|\Delta_j| \in [1-\beta, 1+\beta]$, $j = 1, 2, \dots$. Set $u := 1+\beta$. Assume that $|q| \leq a$, $a \in (0, 0.108]$. Under these assumptions we give an estimation of the moduli of the entries of the matrix $L := L_2^{-1}L_3^{-1}L_4^{-1}\cdots$ that are below the main diagonal. The above lemma implies our next lemma:

Lemma 4.3 *For $\nu < \mu \leq s$ one has $|L_{\mu,\nu}| \leq u^{\mu-\nu}a^{(\mu-\nu)(s-\mu+1)+(\mu-\nu)(\mu-\nu-1)/2}$.*

To obtain a majoration for the entries of L in its sth row one can

1) ignore the presence of the factors $L_2^{-1}, \ldots, L_{s-1}^{-1}$ (because their sth rows have just 1 in position s and zeros elsewhere) and

2) ignore the rows of the matrices L_{s+1}^{-1}, L_{s+2}^{-1}, $\ldots$ below the sth one. Therefore in what follows, instead of L we consider the $s \times s$-matrix $K := \tilde{L}_s\tilde{L}_{s+1}\tilde{L}_{s+2}\cdots$, where $\tilde{L}_j$ is the left upper $s \times s$-minor of the matrix L_j^{-1} ($j \geq s$).

Denote by M an $s\times s$-matrix having on its first subdiagonal the entries $a^{s-1}u$, $a^{s-2}u$, $\ldots$, au and zeros elsewhere. It is clear that $M^s = 0$ and that the nonzero entries of M are majorations of the moduli of the respective entries of the left upper $s \times s$-minor of the matrix N_s. Hence the moduli of the entries of the matrix $\tilde{L}_s$ are majorized by the entries of the matrix $I + \sum_{j=1}^{s-1} M^j$. In the same way the moduli of the entries of the matrix $\tilde{L}_k$, $k \geq s$, are majorized by the entries of the matrix $I+\sum_{j=1}^{s-1}(a^{k-s}M)^j$. Therefore the moduli of the entries of the sth row of K (hence of L as well) are majorized by the entries of the sth row of the product

$$\Pi(a, M) := \prod_{k=s}^{\infty}\left(I + \sum_{j=1}^{s-1}(a^{k-s}M)^j\right)$$

which (taking into account that $M^s = 0$) we represent in the form $I + b_1M + b_2M^2 + \cdots + b_{s-1}M^{s-1}$.

Lemma 4.4 *One has $b_j \leq 1/(1-a)(1-a^2)\cdots(1-a^j)$, $j = 1,\ldots,s-1$.*

The lemma is proved in the next section. The Lemmas 4.2–4.4 imply the inequality (where $1 \leq \nu \leq s-1$)

$$|L_{s,\nu}| \leq \frac{u^{s-\nu}a^{(s-\nu)+(s-\nu)(s-\nu-1)/2}}{(1-a)(1-a^2)\cdots(1-a^{s-\nu})} = \frac{u^{s-\nu}a^{(s-\nu)(s-\nu+1)/2}}{(1-a)(1-a^2)\cdots(1-a^{s-\nu})}\ .$$

Hence the equation (5) for V implies that

$$1 - \sum_{\nu=1}^{s-1}|L_{s,\nu}| \leq |\Delta_1\cdots\Delta_s| \leq 1 + \sum_{\nu=1}^{s-1}|L_{s,\nu}|\ .$$

The following two inequalities (resulting from the conditions $a \in (0, 0.108]$ and $\beta \in (0, 0.7882]$) will be used in our estimates:

$$0 < a < 1/3 \quad \text{and} \quad 0 < au < 1\ . \tag{6}$$

Hence for $\nu = s-1$ (resp. $\nu = s-2$) one has $|L_{s,s-1}| \leq ua/(1-a)$ (resp. $|L_{s,s-2}| = u^2a^2(a/(1-a)(1-a^2)) \leq u^2a^2((1/3)/(2/3)(8/9)) < u^2a^2$). For $\nu \leq s-3$ it is true that

$$\frac{a^{(s-\nu)(s-\nu-1)/2}}{(1-a)(1-a^2)\cdots(1-a^{s-\nu})} = a^{(s-\nu)(s-\nu-3)/2}\left(\frac{a}{1-a}\right)\left(\frac{a}{1-a^2}\right)\cdots\left(\frac{a}{1-a^{s-\nu}}\right) < 1$$

because $a \in (0,1)$ (hence $a^{(s-\nu)(s-\nu-3)/2} \in (0,1)$) and all fractions belong to $(0,1)$ due to $a \in (0, 1/3)$. Hence for $\nu \leq s-2$ the inequalities

$$|L_{s,\nu}| \leq (ua)^{s-\nu} \ \text{ and } \ \sum_{\nu=1}^{s-2}|L_{s,\nu}| \leq (ua)^2 + (ua)^3 + \cdots = (ua)^2/(1-ua)$$

hold true. Thus

$$1-ua/(1-a)-(ua)^2/(1-ua)\leq|\Delta_1\cdots\Delta_s|\leq 1+ua/(1-a)+(ua)^2/(1-ua)\ .$$

Observe that the right-hand and left-hand sides do not depend on s.

We want to choose a and u such that for any s one would have

$$1-\beta/3\leq|\Delta_1\cdots\Delta_s|\leq 1+\beta/3 \quad \text{(recall that } \ u=1+\beta)\ . \tag{7}$$

As $\Delta_s=(\Delta_1\cdots\Delta_s)/(\Delta_1\cdots\Delta_{s-1})$, this implies

$$1-\beta\leq\frac{1-\beta/3}{1+\beta/3}\leq|\Delta_s|\leq\frac{1+\beta/3}{1-\beta/3}\leq 1+\beta\ . \tag{8}$$

The leftmost and rightmost inequalities are true for $\beta\in(0,1)$. Conditions (7) are fulfilled if

$$ua/(1-a)+(ua)^2/(1-ua)\leq(u-1)/3 \tag{9}$$

which is true (together with conditions (6)) for $a=0.108$ and $u=1.7882$.

Now we can finish the proof of the convergence. Recall that the equations $(E_{j,i})$ were defined in the previous section. We represented the variables Δ_j as FPS in q by iterating infinitely many times the following operation: a variable Δ_s is represented in the form $1-qT_{s,r}(q,\Delta^r)$ using the equation $(E_{s,r})$ and then $1-qT_{s,r}(q,\Delta^r)$ is substituted for Δ_s in all other equations of the infinite system. At any step we suppose that $|q|\leq a$ and $|\Delta_j|\in[1-\beta,1+\beta]$. The inequalities given in equation (8) imply that $|1-qT_{s,r}(q,\Delta^r)|\in[1-\beta,1+\beta]$. We finally conclude that the series converge for $|q|\leq a$ and for all such q one has $|\Delta_j|\in[1-\beta,1+\beta]$.

For $a=0.108$, $\beta=0.7882$ one has

$$|\Delta_{j+1}q^{j+1}|\leq(1+\beta)a|q^j|=(1.7882)(0.108)|q|^j<(0.2118)|q|^j=(1-\beta)|q|^j\leq|\Delta_jq^j|$$

which implies that all zeros of $\theta(q,.)$ are distinct. $\square$

5 Proof of Lemmas 4.2 and 4.4

Proof of Lemma 4.2:

The second line of the formula is evident. Unless $\nu\leq\mu\leq s$ the claim of the lemma is trivial. It is also clear that

$$(L_s^{-1})_{\mu,\nu}=((-N_s)^{\mu-\nu})_{\mu,\nu}=(-1)^{\mu-\nu}\Delta_s^{\mu-\nu}((N_s/\Delta_s)^{\mu-\nu})_{\mu,\nu}\ .$$

Set $P_s:=N_s/\Delta_s$. There remains to be proved that

$$((P_s)^{\mu-\nu})_{\mu,\nu}=q^{(\mu-\nu)(s-\mu+1)+(\mu-\nu)(\mu-\nu-1)/2} \quad \text{for} \quad \nu\leq\mu\leq s\ .$$

For $\mu-\nu=1$ this follows from the definition of N_s. Suppose that the above equation holds for $\mu-\nu\leq\kappa$. Note that

$$((P_s)^{\mu-\nu})_{\mu,\nu}=(P_s)_{\mu,\mu-1}((P_s)^{\mu-\nu-1})_{\mu-1,\nu}\ .$$

By induction the right-hand side equals

$$q^{s-\mu+1}q^{(\mu-\nu-1)(s-\mu+2)+(\mu-\nu-1)(\mu-\nu-2)/2}=q^{(\mu-\nu)(s-\mu+1)+(\mu-\nu)(\mu-\nu-1)/2}$$

and the proof follows. $\square$

Proof of Lemma 4.4:

The coefficient b_1 equals $1 + a + a^2 + a^3 + \cdots = 1/(1-a)$ (independent of s). Suppose that the lemma is proved for $j \leq j_0 < s-1$. Set $\Pi_1(a, M) := \prod_{k=s+1}^{\infty}(I + \sum_{j=1}^{s-1}(a^{k-s}M)^j)$ and present Π_1 in the form $I + c_1M + c_2M^2 + \cdots + c_{s-1}M^{s-1}$. It is clear that $\Pi_1(a, M) = \Pi(a, aM)$. Therefore $c_j = b_j a^j$. In what follows we set $b_0 = c_0 = 1$. Hence for $j \leq j_0$ one has $c_j \leq a^j/(1-a)(1-a^2)\cdots(1-a^j)$. Since

$$\Pi(a, M) = (I + M + \cdots + M^{s-1})(I + c_1M + c_2M^2 + \cdots + c_{s-1}M^{s-1}) \ ,$$

one can obtain a term M^{j_0+1} in one of the following ways:

1) one multiplies a term M^{j_0+1} from one of the factors $I + \sum_{j=1}^{s-1}(a^{k-s}M)^j$ $(k \geq s)$ by the terms I of all the others; the sum of all these coefficients equals $\sum_{\nu=0}^{\infty} a^{\nu(j_0+1)} = 1/(1-a^{j_0+1})$;

2) one multiplies M^i from the factor $I + M + \cdots + M^{s-1}$ by $c_{j_0+1-i}M^{j_0+1-i}$ from the second factor for $i = 1, \ldots, j_0$. Thus

$$\begin{array}{rcl} b_{j_0+1} & = & 1/(1-a^{j_0+1}) + c_1 + \cdots + c_{j_0} \\ & \leq & 1/(1-a^{j_0+1}) + \sum_{j=1}^{j_0} a^j/(1-a)(1-a^2)\cdots(1-a^j) \\ & = & 1/(1-a^{j_0+1}) + \sum_{j=1}^{j_0}(1/(1-a)(1-a^2)\cdots(1-a^j) - 1/(1-a)(1-a^2)\cdots(1-a^{j-1})) \\ & = & 1/(1-a^{j_0+1}) - 1 + 1/B \\ & = & a^{j_0+1}/(1-a^{j_0+1}) + 1/B \ , \end{array}$$

where $B = (1-a)(1-a^2)\cdots(1-a^{j_0})$. One has

$$a^{j_0+1}/(1-a^{j_0+1}) + 1/B = (a^{j_0+1}(B-1)+1)/(1-a^{j_0+1})B < 1/(1-a^{j_0+1})B$$

because $B \in (0, 1)$. This proves the lemma by induction on j. $\square$

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